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Ques:- What do you mean by Fourier Series? Evaluate Fourier Co-efficients?

Ans

Fourier Series :-

Any finite continuous single valued periodic function can be expressed as a summation of simple harmonic terms having frequencies which are multiples of that of the given function.

For Example :→ A periodic function $f(x)$ is defined in interval $(-\pi, \pi)$ i.e for $-\pi < x < \pi$ and having a period 2π can be expressed as

$$f(x) = A_0 + \sum_{r=1}^{\infty} (A_r \cos rx + B_r \sin rx)$$

or

$$= A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_r \cos rx + \cdots$$
$$\cdots B_1 \sin x + B_2 \sin 2x + \cdots + B_r \sin rx + \cdots \quad (1)$$

This is known as "Fourier series".

The constants A_0, A_r & B_r are known as "Fourier co-efficients".

Evaluation of A_0 :-

Multiplying by eqn (1) by dx and integrate eqn (1) from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)dx = A_0 \int_{-\pi}^{\pi} dx + A_1 \int_{-\pi}^{\pi} \cos x\, dx + \cdots + A_r \int_{-\pi}^{\pi} \cos rx\, dx + \cdots$$
$$+ B_1 \int_{-\pi}^{\pi} \sin x\, dx + \cdots + B_r \int_{-\pi}^{\pi} \sin rx\, dx + \cdots$$
$$= A_0 [x]_{-\pi}^{\pi}$$
$$= 2\pi A_0$$

$$\text{or } A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad \cdots\cdots \quad (2)$$

EVALUATION FOR A_r :→

Multiplying by eqn (1) by $rx dx$ and integrating from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\cos rx\,dx = A_0\int_{-\pi}^{\pi}\cos rx\,dx + A_1\int_{-\pi}^{\pi}\cos x\cdot\cos rx\,dx + \cdots$$

$$+ \int_{-\pi}^{\pi}\cos rx\cdot\cos rx\,dx + \cdots + B_1\int_{-\pi}^{\pi}\sin x\cdot\cos rx\,dx$$

$$+ \cdots + B_r\int_{-\pi}^{\pi}\sin rx\cdot\cos rx\,dx + \cdots$$

$$= A_r\int_{-\pi}^{\pi}\cos^2 rx\,dx$$

$$= A_r\int_{-\pi}^{\pi}\frac{2\cos^2 rx}{2}\,dx = \frac{A_r}{2}\int_{-\pi}^{\pi}(1+\cos 2rx)\,dx$$

$$= \frac{A_r}{2}\left[x + \frac{\sin 2rx}{2r}\right]_{-\pi}^{\pi}$$

$$= \frac{A_r}{2}\left[\pi + \frac{\sin 2r\pi}{2r} - (-\pi) - \frac{\sin(-2r\pi)}{2r}\right]$$

$$= \frac{A_r}{2}\left[\pi+\pi\right]$$

$$= \cancel{2}\pi \times \frac{A_r}{\cancel{2}}$$

$$= \pi\cdot A_r$$

$$\text{or } A_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\,dx \quad \text{-------} \quad (3)$$

EVALATION OF B_r :- multiplying (1) by rx and integrating from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\sin rx\,dx = A_0\int_{-\pi}^{\pi}\sin rx\,dx + A_1\int_{-\pi}^{\pi}\cos x\sin rx\,dx + \cdots +$$

$$A_r\int_{-\pi}^{\pi}\cos rx\sin rx\,dx + \cdots + B_1\int_{-\pi}^{\pi}\sin x\cdot\sin rx\,dx$$

$$+ \cdots + B_r\int_{-\pi}^{\pi}\sin rx\cdot\sin rx\,dx$$

$$= B_r\int_{-\pi}^{\pi}\sin^2 rx\,dx$$

$$= \frac{B_r}{2}\int_{-\pi}^{\pi}2\sin^2 rx\,dx$$

$$= \frac{B_r}{2}\int_{-\pi}^{\pi}(1-\cos 2rx)\,dx$$

$$= \frac{B_r}{2}\left[\pi - \frac{\sin 2rx}{2r}\right]_{-\pi}^{\pi}$$

$$= \frac{B_r}{2}\left[\pi - \frac{\sin 2r\pi}{2r} - (-\pi) + \frac{\sin 2r\pi}{2r}\right]$$

$$= \pi \cdot B_r$$

$$\therefore B_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin rx \, dx$$

Because of the periodicity of the integrands, the interval of integration in equation (2), (3) & (4) may be replaced by other interval of length 2π. For instant, by the interval 0 to 2π.